

The Stokes phenomenon for the q -difference equation satisfied by the basic hypergeometric series ${}_3\varphi_1(a_1, a_2, a_3; b_1; q, x)$

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Abstract

We show the connection formula for the basic hypergeometric series ${}_3\varphi_1(a_1, a_2, a_3; b_1; q, x)$ between around the origin and infinity by the using of the q -Borel-Laplace transformations. We also show the limit $q \rightarrow 1 - 0$ of the new connection formula.

1 Introduction

In this paper, we show the connection formula for the *divergent* basic hypergeometric series

$${}_3\varphi_1(a_1, a_2, a_3; b_1; q, x) = \sum_{n \geq 0} \frac{(a_1, a_2, a_3; q)_n}{(b_1; q)_n (q; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{-1} x^n \quad (1)$$

between around the origin and around infinity by the using of the q -Borel-Laplace resummation methods. Here, the function $(a; q)_n$ is the q -shifted factorial (see section 2 and [4] for more details of the q -shifted factorials and the basic hypergeometric series ${}_r\varphi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, x)$):

$$(a; q)_n := \begin{cases} 1, & n = 0, \\ (1 - a)(1 - aq) \dots (1 - aq^{n-1}), & n \geq 1. \end{cases}$$

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The series (1) satisfy the third order linear q -difference equation

$$\begin{aligned} \left(a_1 a_2 a_3 x - \frac{b_1}{q^2}\right) u(q^3 x) - \left\{ (a_1 a_2 + a_2 a_3 + a_3 a_1) x - \left(\frac{b_1}{q^2} + \frac{1}{q}\right) \right\} u(q^2 x) \\ + \left\{ (a_1 + a_2 + a_3) x - \frac{1}{q} \right\} u(qx) - xu(x) = 0. \end{aligned} \quad (2)$$

Equation (2) also has a fundamental system of solutions around infinity:

$$v_1(x) := x^{-\alpha_1} {}_3\varphi_2 \left(a_1, \frac{a_1 q}{b_1}, 0; \frac{a_1 q}{a_2}, \frac{a_1 q}{a_3}; q, \frac{qb_1}{a_1 a_2 a_3 x} \right) \quad (3)$$

$$v_2(x) := x^{-\alpha_2} {}_3\varphi_2 \left(a_2, \frac{a_2 q}{b_1}, 0; \frac{a_2 q}{a_1}, \frac{a_2 q}{a_3}; q, \frac{qb_1}{a_1 a_2 a_3 x} \right) \quad (4)$$

$$v_3(x) := x^{-\alpha_3} {}_3\varphi_2 \left(a_3, \frac{a_3 q}{b_1}, 0; \frac{a_3 q}{a_2}, \frac{a_3 q}{a_1}; q, \frac{qb_1}{a_1 a_2 a_3 x} \right) \quad (5)$$

where $a_j = q^{\alpha_j}$, $j = 1, 2$ and 3 . In section 3, we show the connection formula between (3), (4) (5) and (1).

We review the connection problems on the linear q -difference equations. Connection problems on the linear q -difference equations with regular singular points were studied by G. D. Birkhoff [1]. Connection formulae for the second order linear q -difference equations are given by the matrix form

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} C_{11}(x) & C_{12}(x) \\ C_{21}(x) & C_{22}(x) \end{pmatrix} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}.$$

The pair $(u_1(x), u_2(x))$ is a fundamental system of solutions around the origin and the pair $(v_1(x), v_2(x))$ is a fundamental system of solutions around infinity. The connection coefficients $C_{jk}(x)$ ($1 \leq j, k \leq 2$) are given by q -periodic and unique valued functions

$$\sigma_q C_{jk}(x) = C_{jk}(x), \quad C_{jk}(e^{2\pi i} x) = C_{jk}(x),$$

namely, the *elliptic functions*.

The first example of the connection formula was given by G. N. Watson [11] in 1910. Watson gave the connection formula for Heine's basic hypergeometric series

$${}_2\varphi_1(a, b; c; q, x) := \sum_{n \geq 0} \frac{(a, b; q)_n}{(c; q)_n (q; q)_n} x^n$$

around the origin and around the infinity [4, page 117]. Heine's ${}_2\varphi_1(a, b; c; q, x)$ satisfies the q -difference equation

$$[(c - abqx)\sigma_q^2 - \{(c + q) - (a + b)qx\} \sigma_q + q(1 - x)] u(x) = 0. \quad (6)$$

The equation (6) also has a fundamental system of solutions around the infinity:

$$y_\infty^{(a,b)}(x) = x^{-\alpha} {}_2\varphi_1\left(a, \frac{aq}{c}; \frac{aq}{b}; q, \frac{cq}{abx}\right)$$

and

$$y_\infty^{(b,a)}(x) = x^{-\beta} {}_2\varphi_1\left(b, \frac{bq}{c}; \frac{bq}{a}; q, \frac{cq}{abx}\right),$$

provided that $a = q^\alpha$ and $b = q^\beta$. Watson's connection formula for ${}_2\varphi_1(a, b; c; q, x)$ is given by

$$\begin{aligned} {}_2\varphi_1(a, b; c; q; x) &= \frac{(b, c/a; q)_\infty \theta(-ax)_\infty}{(c, b/a; q)_\infty \theta(-x)_\infty} \frac{\theta(x)}{\theta(ax)} y_\infty^{(a,b)}(x) \\ &+ \frac{(a, c/b; q)_\infty \theta(-bx)_\infty}{(c, a/b; q)_\infty \theta(-x)_\infty} \frac{\theta(x)}{\theta(bx)} y_\infty^{(b,a)}(x). \end{aligned}$$

Here, the notation $\theta(x)$ is the theta function of Jacobi (see section two for more details). We remark that the connection coefficients are given by the q -elliptic functions.

But connection formulae for q -difference equations with irregular singular points had not known for a long time. We remark that A. Duval and C. Mitschi gave connection matrices for degenerated *differential* equations [3]. The irregularity of q -difference equations are studied by the using of the Newton polygons by J.-P. Ramis, J. Sauloy and C. Zhang [9]. C. Zhang gave connection formulae for some confluent type basic hypergeometric series [12, 13, 14] where he uses the q -Borel-Laplace transformations. In [6, 7], the author gave the connection formula for the Hahn-Exton q -Bessel function and the q -confluent type function by the q -Borel-Laplace transformations. These resummation methods are powerful tools for connection problems on linear q -difference equations with irregular singular points.

Definition 1. We assume that $f(x)$ is a formal power series $f(x) = \sum_{n \in \mathbb{Z}} a_n x^n$, $a_0 = 1$.

1. The q -Borel transformation is

$$(\mathcal{B}_q^+ f)(\xi) := \sum_{n \in \mathbb{Z}} a_n q^{\frac{n(n-1)}{2}} \xi^n (=:\psi(\xi)).$$

2. For any analytic function $\psi(\xi)$ around $\xi = 0$, the q -Laplace transformation is

$$(\mathcal{L}_{q,\lambda}^+ \psi)(x) := \frac{1}{1-q} \int_0^{\lambda\infty} \frac{\varphi(\xi)}{\theta_q\left(\frac{\xi}{x}\right)} \frac{d_q \xi}{\xi} = \sum_{n \in \mathbb{Z}} \frac{\varphi(\lambda q^n)}{\theta_q\left(\frac{\lambda q^n}{x}\right)}.$$

Here, this transformation is given by Jackson's q -integral [4, page 23].

The definition is a special case of one of the q -Laplace transformations in [2, 12]. The q -Borel transformation is the formal inverse of the q -Laplace transformation as follows:

Lemma 1 (Zhang, [12]). *For any entire function $f(x)$, we have*

$$\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ f = f.$$

Thanks to these methods, some connection formulae for the second order q -difference equations were found. However, the connection formulae for more higher order linear q -difference equations have not known. In this paper, especially we apply the q -Borel-Laplace transformations to the divergent series (1) to study the connection problem on the third order q -difference equation. In the section 3, we show the following theorem:

Theorem. *For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have*

$$\begin{aligned} {}_3f_1(a_1, a_2, a_3; b_1; q; \lambda, x) &:= (\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ {}_3\varphi_1(a_1, a_2, a_3; b_1; q, x))(x) \\ &= \frac{(a_2, a_3, b_1/a_1; q)_\infty}{(b_1, a_2/a_1, a_3/a_1; q)_\infty} \frac{\theta(a_1\lambda)}{\theta(\lambda)} \frac{\theta(a_1qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_1x)} v_1(x) \\ &+ \frac{(a_1, a_3, b_1/a_2; q)_\infty}{(b_1, a_1/a_2, a_3/a_2; q)_\infty} \frac{\theta(a_2\lambda)}{\theta(\lambda)} \frac{\theta(a_2qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_2x)} v_2(x) \\ &+ \frac{(a_2, a_1, b_1/a_3; q)_\infty}{(b_1, a_2/a_3, a_1/a_3; q)_\infty} \frac{\theta(a_3\lambda)}{\theta(\lambda)} \frac{\theta(a_3qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_3x)} v_3(x). \end{aligned}$$

Here, $(\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ {}_3\varphi_1(a_1, a_2, a_3; b_1; q, x))(x)$ is the q -Borel-Laplace transform of the divergent series ${}_3\varphi_1(a_1, a_2, a_3; b_1; q, x)$.

We remark that the connection coefficients (with the new parameter λ) are given by the q -elliptic functions. These coefficients are also the new example of the Stokes phenomenon [2] for the q -difference equation (2).

In the last section, we also give the limit $q \rightarrow 1 - 0$ of the new connection formula.

2 Basic notations

In this section, we review our notations. The q -shifted operator σ_q is given by $\sigma_q f(x) = f(qx)$. For any fixed $\lambda \in \mathbb{C}^* \setminus q^{\mathbb{Z}}$, the set $[\lambda; q]$ -spiral is $[\lambda; q] := \lambda q^{\mathbb{Z}} = \{\lambda q^k; k \in \mathbb{Z}\}$. The function $(a; q)_n$ is the q -shifted factorial such that

$$(a; q)_n := \begin{cases} 1, & n = 0, \\ (1 - a)(1 - aq) \dots (1 - aq^{n-1}), & n \geq 1. \end{cases}$$

moreover, $(a; q)_\infty := \lim_{n \rightarrow \infty} (a; q)_n$ and

$$(a_1, a_2, \dots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \dots (a_m; q)_\infty.$$

The basic hypergeometric series with the base q [4, page 4] is

$$\begin{aligned} {}_r\varphi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, x) \\ := \sum_{n \geq 0} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n (q; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{1+s-r} x^n. \end{aligned}$$

The radius of convergence is $\infty, 1$ or 0 according to whether $r - s < 1, r - s = 1$ or $r - s > 1$.

The theta function of Jacobi is important in connection problems on linear q -difference equations. The theta function with the base q is

$$\theta_q(x) := \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n, \quad \forall x \in \mathbb{C}^*.$$

The theta function has the triple product identity

$$\theta_q(x) = \left(q, -x, -\frac{q}{x}; q \right)_\infty. \quad (7)$$

The theta function satisfies the q -difference equation $\theta_q(q^k x) = q^{-\frac{n(n-1)}{2}} x^{-k} \theta_q(x)$, $\forall k \in \mathbb{Z}$. The theta function also has the inversion formula $\theta_q(1/x) = \theta_q(x)/x$.

We remark that $\theta(\lambda q^k/x) = 0$ if and only if $x \in [-\lambda; q]$. The function $\theta(x)/\theta(q^\alpha x)$, $\forall \alpha \notin \mathbb{Z}$ satisfies a q -difference equation

$$u(qx) = q^\alpha u(x),$$

which is also satisfied by the function $u(x) = x^\alpha$.

3 The connection formula

In this section, we give the new connection formula for the basic hypergeometric series ${}_3\varphi_1(a_1, a_2, a_3; b_1; q, x)$. In section 3.1, we review the connection formula of non-degenerated series ${}_3\varphi_2(a_1, a_2, a_3; b_1, b_2; q, x)$.

3.1 The non-degenerated case

The non-degenerated convergent series

$${}_3\varphi_2(a_1, a_2, a_3; b_1, b_2; q, x) := \sum_{n \geq 0} \frac{(a_1, a_2, a_3; q)_n}{(b_1, b_2; q)_n (q; q)_n} x^n \quad (8)$$

satisfies the third order q -difference equation

$$\left[\left(a_1 a_2 a_3 x - \frac{b_1 b_2}{q^2} \right) \sigma_q^3 - \left\{ (a_1 a_2 + a_2 a_3 + a_3 a_1) x - \left(\frac{b_1 b_2}{q^2} + \frac{b_2}{q} + \frac{b_1}{q} \right) \right\} \sigma_q^2 \right. \\ \left. - \left\{ (a_1 + a_2 + a_3) x - \left(\frac{b_1}{q} + \frac{b_2}{q} + 1 \right) \right\} \sigma_q - (x - 1) \right] u(x) = 0. \quad (9)$$

Equation (9) also has a fundamental system of solutions around infinity:

$$\tilde{v}_1(x) = \frac{\theta(a_1 x)}{\theta(x)} {}_3\varphi_2 \left(a_1, \frac{a_1 q}{b_1}, \frac{a_1 q}{b_2}; \frac{a_1 q}{a_2}, \frac{a_1 q}{a_3}; q, \frac{q b_1 b_2}{a_1 a_2 a_3 x} \right), \quad (10)$$

$$\tilde{v}_2(x) = \frac{\theta(a_2 x)}{\theta(x)} {}_3\varphi_2 \left(a_2, \frac{a_2 q}{b_1}, \frac{a_2 q}{b_2}; \frac{a_2 q}{a_1}, \frac{a_2 q}{a_3}; q, \frac{q b_1 b_2}{a_1 a_2 a_3 x} \right), \quad (11)$$

$$\tilde{v}_3(x) = \frac{\theta(a_3 x)}{\theta(x)} {}_3\varphi_2 \left(a_3, \frac{a_3 q}{b_1}, \frac{a_3 q}{b_2}; \frac{a_3 q}{a_2}, \frac{a_3 q}{a_1}; q, \frac{q b_1 b_2}{a_1 a_2 a_3 x} \right). \quad (12)$$

The connection formula between the solutions (10), (11), (12) and (8) can be found in [4, page 121]. We remark that the following formula was essentially given by L. J. Slater.

Theorem 1 (Slater, [10]). *For any $x \in \mathbb{C}^*$, we have*

$$\begin{aligned} {}_3\varphi_2(a_1, a_2, a_3; b_1, b_2; q, x) &= \frac{(a_2, a_3, b_1/a_1, b_2/a_1; q)_\infty}{(b_1, b_2, a_2/a_1, a_3/a_1; q)_\infty} \frac{\theta(-a_1x)}{\theta(-x)} \frac{\theta(x)}{\theta(a_1x)} \tilde{v}_1 \\ &\quad + \text{idem}(a_1; a_2, a_3). \end{aligned}$$

Provided that the notation $\text{idem}(a_1; a_2, a_3)$ after an expression stands for the sum expressions obtained from the preceding expression by interchanging a_1 with each a_2 and a_3 .

This Theorem can be considered as the higher order extension of Watson's formula. By Theorem 1, we obtain the following key Lemma.

Lemma 2. *For any $x \in \mathbb{C}^*$, we have*

$$\begin{aligned} {}_3\varphi_2(a_1, a_2, a_3; b_1, 0; q, x) &= \frac{(a_2, a_3, b_1/a_1; q)_\infty}{(b_1, a_2/a_1, a_3/a_1; q)_\infty} \frac{\theta(-a_1x)}{\theta(-x)} {}_2\varphi_2\left(a_1, \frac{a_1q}{b_1}; \frac{a_1q}{a_2}, \frac{a_1q}{a_3}; q, \frac{q^2b_1}{a_2a_3x}\right) \\ &\quad + \text{idem}(a_1; a_2, a_3). \end{aligned}$$

Proof. We take the limit $b_2 \rightarrow 0$ in Theorem 1, we obtain the conclusion. \square

In the next section, we prove our new connection formula by Lemma 2 and the q -Borel-Laplace transformations.

3.2 Proof of main Theorem

In this section, we prove the following Theorem.

Theorem 2. *For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have*

$$\begin{aligned} {}_3f_1(a_1, a_2, a_3; b_1; q; \lambda, x) &:= (\mathcal{L}_{q, \lambda}^+ \circ \mathcal{B}_q^+ {}_3\varphi_1(a_1, a_2, a_3; b_1; q, x))(x) \\ &= \frac{(a_2, a_3, b_1/a_1; q)_\infty}{(b_1, a_2/a_1, a_3/a_1; q)_\infty} \frac{\theta(a_1\lambda)}{\theta(\lambda)} \frac{\theta(a_1qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_1x)} v_1(x) \\ &\quad + \frac{(a_1, a_3, b_1/a_2; q)_\infty}{(b_1, a_1/a_2, a_3/a_2; q)_\infty} \frac{\theta(a_2\lambda)}{\theta(\lambda)} \frac{\theta(a_2qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_2x)} v_2(x) \\ &\quad + \frac{(a_2, a_1, b_1/a_3; q)_\infty}{(b_1, a_2/a_3, a_1/a_3; q)_\infty} \frac{\theta(a_3\lambda)}{\theta(\lambda)} \frac{\theta(a_3qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_3x)} v_3(x). \end{aligned}$$

Proof. We apply the q -Borel transformation to the series ${}_3\varphi_1(a_1, a_2, a_3; b_1; q, x)$.

$$(\mathcal{B}_q^+ {}_3\varphi_1(a_1, a_2, a_3; b_1; q, x))(\xi) = {}_3\varphi_2(a_1, a_2, a_3; b_1, 0, -\xi) =: \varphi(\xi).$$

By Lemma 2, we have another expression of the function $\varphi(\xi)$. We also apply the q -Laplace transformation $\mathcal{L}_{q,\lambda}$ to the function $\varphi(\xi)$, we obtain the conclusion. \square

Remark 1. We remark that the fundamental system of solutions for equation (2) are given by

$$v_1(x) := \frac{\theta(a_1x)}{\theta(x)} {}_3\varphi_2\left(a_1, \frac{a_1q}{b_1}, 0; \frac{a_1q}{a_2}, \frac{a_1q}{a_3}; q, \frac{qb_1}{a_1a_2a_3x}\right), \quad (13)$$

$$v_2(x) := \frac{\theta(a_2x)}{\theta(x)} {}_3\varphi_2\left(a_2, \frac{a_2q}{b_1}, 0; \frac{a_2q}{a_1}, \frac{a_2q}{a_3}; q, \frac{qb_1}{a_1a_2a_3x}\right), \quad (14)$$

$$v_3(x) := \frac{\theta(a_3x)}{\theta(x)} {}_3\varphi_2\left(a_3, \frac{a_3q}{b_1}, 0; \frac{a_3q}{a_2}, \frac{a_3q}{a_1}; q, \frac{qb_1}{a_1a_2a_3x}\right) \quad (15)$$

in the Theorem 2.

Remark 2. By the q -difference equation of the theta function, we can check out that the connection coefficients (with the new parameter λ)

$$\begin{aligned} C_1(x) &:= \frac{(a_2, a_3, b_1/a_1; q)_\infty}{(b_1, a_2/a_1, a_3/a_1; q)_\infty} \frac{\theta(a_1\lambda)}{\theta(\lambda)} \frac{\theta(a_1qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_1x)}, \\ C_2(x) &:= \frac{(a_1, a_3, b_1/a_2; q)_\infty}{(b_1, a_1/a_2, a_3/a_2; q)_\infty} \frac{\theta(a_2\lambda)}{\theta(\lambda)} \frac{\theta(a_2qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_2x)}, \\ C_3(x) &:= \frac{(a_2, a_1, b_1/a_3; q)_\infty}{(b_1, a_2/a_3, a_1/a_3; q)_\infty} \frac{\theta(a_3\lambda)}{\theta(\lambda)} \frac{\theta(a_3qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_3x)} \end{aligned}$$

are the q -elliptic functions.

4 The limit $q \rightarrow 1 - 0$ of the connection formula

The aim of this section is to give the limit $q \rightarrow 1 - 0$ of the new connection formula as follows:

Theorem 3. For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have the following limit $q \rightarrow 1-0$ of the connection formula

$$\begin{aligned} & \lim_{q \rightarrow 1-0} {}_3f_1(q^{\alpha_1}, q^{\alpha_2}, q^{\alpha_3}; q^{\beta_1}; q; \lambda, x) \\ &= \frac{\Gamma(\beta_1)\Gamma(\alpha_2 - \alpha_1)\Gamma(\alpha_3 - \alpha_1)}{\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\beta_1 - \alpha_1)} x^{-\alpha_1} {}_2F_2\left(\alpha_1, \alpha_1 + 1 - \beta_1; \alpha_1 + 1 - \alpha_2, \alpha_1 + 1 - \alpha_3; \frac{1}{x}\right) \\ &+ \frac{\Gamma(\beta_1)\Gamma(\alpha_1 - \alpha_2)\Gamma(\alpha_3 - \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_3)\Gamma(\beta_1 - \alpha_2)} x^{-\alpha_2} {}_2F_2\left(\alpha_2, \alpha_2 + 1 - \beta_1; \alpha_2 + 1 - \alpha_1, \alpha_2 + 1 - \alpha_3; \frac{1}{x}\right) \\ &+ \frac{\Gamma(\beta_1)\Gamma(\alpha_2 - \alpha_3)\Gamma(\alpha_1 - \alpha_3)}{\Gamma(\alpha_2)\Gamma(\alpha_1)\Gamma(\beta_1 - \alpha_3)} x^{-\alpha_3} {}_2F_2\left(\alpha_3, \alpha_3 + 1 - \beta_1; \alpha_3 + 1 - \alpha_2, \alpha_3 + 1 - \alpha_1; \frac{1}{x}\right), \end{aligned}$$

provided that $-\pi < \arg x < \pi$.

The following proposition [13] is important to consider the limit $q \rightarrow 1-0$ of our connection formula.

Proposition 1. For any $x \in \mathbb{C}^* (-\pi < \arg x < \pi)$, we have

$$\lim_{q \rightarrow 1-0} \frac{\theta(q^\beta x)}{\theta(q^\alpha x)} = x^{\alpha-\beta} \quad (16)$$

and

$$\lim_{q \rightarrow 1-0} \frac{\theta\left(\frac{q^\alpha x}{(1-q)}\right)}{\theta\left(\frac{q^\beta x}{(1-q)}\right)} (1-q)^{\beta-\alpha} = x^{\beta-\alpha}. \quad (17)$$

We also review the q -gamma function. The q -gamma function $\Gamma_q(x)$ is

$$\Gamma_q(x) := \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x}, \quad 0 < q < 1.$$

The limit $q \rightarrow 1-0$ of $\Gamma_q(x)$ gives the gamma function [4, page 20]

$$\lim_{q \rightarrow 1-0} \Gamma_q(x) = \Gamma(x). \quad (18)$$

We give the proof of the Theorem 3.

Proof. At first, we put $a_j := q^{\alpha_j}$ ($j = 1, 2, 3$), $b_1 := q^{\beta_1}$ and $x \mapsto x/(1 - q)$. We remark that the limit $q \rightarrow 1 - 0$ of the left hand-side of Theorem 3 formally converges the hypergeometric series

$${}_3F_1(\alpha_1, \alpha_2, \alpha_3; \beta_1; x) = \sum_{n \geq 0} \frac{(\alpha_1, \alpha_2, \alpha_3)_n}{(\beta_1)_n n!} x^n.$$

We consider the right hand-side. The connection formula can be rewritten as follows:

$$\begin{aligned} & {}_3f_1(q^{\alpha_1}, q^{\alpha_2}, q^{\alpha_3}; q^{\beta_1}; q; \lambda, x) \\ &= \frac{(q^{\alpha_2}, q^{\alpha_3}, q^{\beta_1 - \alpha_1}; q)_\infty}{(q^{\beta_1}, q^{\alpha_2 - \alpha_1}, q^{\alpha_3 - \alpha_1}; q)_\infty} \frac{\theta(q^{\alpha_1} \lambda)}{\theta(\lambda)} \frac{\theta\left(\frac{q^{\alpha_1 + 1} x}{\lambda(1 - q)}\right)}{\theta\left(\frac{qx}{\lambda(1 - q)}\right)} \\ & \times {}_3\varphi_2\left(q^{\alpha_1}, q^{\alpha_1 + 1 - \beta_1}, 0; q^{\alpha_1 + 1 - \alpha_2}, q^{\alpha_1 + 1 - \alpha_3}; q, \frac{q^{1 + \beta_1}(1 - q)}{q^{\alpha_1 + \alpha_2 + \alpha_3} x}\right) \\ & + \text{idem}(q^{\alpha_1}; q^{\alpha_2}, q^{\alpha_3}) \\ &= \frac{\Gamma_q(\beta_1) \Gamma_q(\alpha_2 - \alpha_1) \Gamma_q(\alpha_3 - \alpha_1)}{\Gamma_q(\alpha_2) \Gamma_q(\alpha_3) \Gamma_q(\beta_1 - \alpha_1)} \frac{\theta(q^{\alpha_1} \lambda)}{\theta(\lambda)} \left\{ \frac{\theta\left(\frac{q^{\alpha_1 + 1} x}{\lambda(1 - q)}\right)}{\theta\left(\frac{qx}{\lambda(1 - q)}\right)} (1 - q)^{-\alpha_1} \right\} \\ & \times {}_3\varphi_2\left(q^{\alpha_1}, q^{\alpha_1 + 1 - \beta_1}, 0; q^{\alpha_1 + 1 - \alpha_2}, q^{\alpha_1 + 1 - \alpha_3}; q, \frac{q^{1 + \beta_1}(1 - q)}{q^{\alpha_1 + \alpha_2 + \alpha_3} x}\right) \\ & + \text{idem}(q^{\alpha_1}; q^{\alpha_2}, q^{\alpha_3}). \end{aligned}$$

By (16), (17) and (18), we obtain the conclusion. \square

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